CHERN CLASSES FOR COHERENT SHEAVES

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INTRODUCTION

We present in this paper a construction of Chern classes for a coherent sheaf S on a complex manifold X. In fact we construct classes $C_p(S)$ in $H^{2p}(X,\mathbb{C})$, depending only on the smooth equivalence class of the sheaf S, and analytic classes $C_p^A(S)$ in $H^p(X,\Omega^p)$ (Ω^p denotes the sheaf of germs of holomorphic p-forms on X). We show that these invariants extend the classical and Atiyah constructions for locally free sheaves, and that the analytic class $C_p^A(S)$ is the 'leading term' of type (p,p) in the expansion by type of $C_p(S)$. In the case of a compact manifold, we are able to show that the classes $C_p(S)$ correspond (up to a constant factor) with those defined by Atiyah and Hirzebruch in [2]. Thus in particular the $C_p(S)$ are integral cohomology classes.

The major obstacle to any global work with coherent analytic sheaves is the lack of a global resolution by locally free analytic sheaves. In [2], Atiyah and Hirzebruch circumvent this problem by using real-analytic functions. It follows from results of Grauert (given in [2]) that on a compact complex manifold, a coherent sheaf has a global resolution by locally free real-analytic sheaves. However, this approach does not enable one to recover analytic information about coherent sheaves. In this paper, we use the notion of 'twisted resolution' developed by Toledo and Tong (see [6, 7, 8, 9, 15]), and the form of equivalence developed by O'Brian (see [9,15]).

A twisted resolution of a coherent sheaf S is a triple $(\mathcal{U},\xi,a)$, whose $\mathcal{U}$ is an open cover, and ξ denotes a collection of local resolutions

$$\xi_\alpha^{\cdot} \to \mathcal{S}|U_\alpha \to 0$$

by locally free sheaves over each open set U_α of $\mathcal{U}$, which roughly speaking are 'glued together' by the twisting cochain a. In this paper we show that the twisted resolution $(\mathcal{U},\xi,a)$ may be used to construct a global resolution on an 'approximation' to the manifold X, the simplicial manifold $X_{\mathcal{U}}^{\cdot}$, where sheaves on $X_{\mathcal{U}}^{\cdot}$ are defined as in [11]. We believe that this construction will provide a natural way to obtain global analytic information about coherent sheaves.

To define the Chern classes from this resolution, we choose local connections for each ξ_α^i. We then show that these connections naturally define a connection on the sheaves of our resolution 'pulled back' to $X_{\mathcal{U}}^p \times |\Delta_p|$ (where $|\Delta_p|$ denotes the geometric p-simplex). We then proceed in the classical manner to apply Chern polynomials to the curvature of this connection, obtaining Chern forms in the 'simplicial De Rham complex' (see [12]). By integration over the fibre, we obtain cocycles in the Čech complex $\mathrm{Tot}(C^*(\mathcal{U},A^*))$ (A^* denotes the De Rham sheaves on X). Finally we show that the 'leading term' of this cocycle in bidegree (p,p) defines a Čech cocycle in the analytic Čech complex $C^*(\mathcal{U},\Omega^p)$.

Whilst this work was in progress, we obtained the preprints [7] of Tong's, [15] of Tong and O'Brian's. In these papers an analytic Chern character is constructed by directly writing down formulae in the local data $(\mathcal{U},\xi,a)$. We intend to prove elsewhere that the analytic Chern classes $c_p^A(\mathcal{S})$ of this paper give rise, by the usual universal formula, to the same Chern character as in the work of Tong and O'Brian. We hope that this will make it possible to unify all the various attempts to construct global invariants for coherent sheaves.

I am indebted to Toledo, Tong and O'Brian for much useful discussion, advice and criticism.

0. TERMINOLOGY AND DEFINITIONS.

Suppose that X is a paracompact complex-analytic manifold.

0.0. Definition.

Let $\mathcal{O}_X$ denote the sheaf of germs of holomorphic functions on X.

Let $\mathcal{R}_X$ denote the sheaf of germs of complex-valued real-analytic functions on X.

Let Σ_X denote the sheaf of germs of complex-valued smooth functions on X.

0.1. Lemma.

The inclusions $\mathcal{O}_X \subset \mathcal{R}_X \subset \Sigma_X$ are flat. That is, $\mathcal{R}_X$ is a flat $\mathcal{O}_X$-module, and Σ_X is a flat $\mathcal{R}_X$-module. Hence Σ_X is a flat $\mathcal{O}_X$-module.

Proof:

See Atiyah-Hirzebruch [2].

0.2. Definition.

Suppose that R is a ring. Let

$$0 \to M_0 \to \ldots \to M_n \to 0$$

be a sequence of R-modules. We will call such a sequence *elementary* if it is a sum of terms of the form

$$0 \to M \xrightarrow{\text{id}} M \to 0.$$

2. O.A. TWISTING COCHAINS (see [6,7,9]).

Suppose that $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ is a locally finite open cover of X, and that over each open set U_α we have a finitary complex of locally free $\mathcal{O}_{U_\alpha}$-modules $(\xi_\alpha^\cdot, a_\alpha)$ with differential a_α of degree +1. Then we can construct a bigraded module over the holomorphic functions on X as follows.

Over the intersection $U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots U_{\alpha_p} = U_{\alpha_0,\ldots\alpha_p}$ we define $\mathrm{Hom}^q(\xi,\xi)|U_{\alpha_0,\ldots\alpha_p}$ to be the maps of degree q

$$\xi_{\alpha_p}^\cdot|U_{\alpha_0\ldots\alpha_p} \to \xi_{\alpha_0}^\cdot|U_{\alpha_0\ldots\alpha_p} .$$

Then we have

O.A.1. Definition.

$$C^p(\mathcal{U},\mathrm{Hom}^q(\xi,\xi)) = \prod_{\substack{(\alpha_0\ldots\alpha_p) \\ \text{s.t.} U_{\alpha_0\ldots\alpha_p}\neq\emptyset}} \mathrm{Hom}^q(\xi,\xi)|_{U_{\alpha_0,\ldots\alpha_p}} .$$

Now we can define a product as follows. If

$$f^{p,q},\ g^{r,s} \in C^*(\mathcal{U},\mathrm{Hom}^*(\xi,\xi))$$

then we define

O.A.2. Definition.

$$(f^{p,q}.\ g^{r,s})_{\alpha_0,\ldots\alpha_{p+r}} = (-1)^{qr} f^{p,q}_{\alpha_0\ldots\alpha_p}\ g^{r,s}_{\alpha_p,\ldots\alpha_{p+r}}$$

This product is clearly associative. We may also define a differential.

0.A.3. Definition.

$$\delta : C^p(\mathcal{U}, \mathrm{Hom}^q(\xi,\xi)) \to C^{p+1}(\mathcal{U}, \mathrm{Hom}^q(\xi,\xi))$$

$$(\delta f)_{\alpha_o,\ldots,\alpha_{p+1}} = \sum_{i=1}^{p} (-1)^i f_{\alpha_o,\ldots\hat{\alpha}_i,\ldots\alpha_{p+1}} |U_{\alpha_o,\ldots,\alpha_{p+1}}$$

0.A.4. Definition.

A *twisting cochain* is an element $a = \sum_{p=0}^{\infty} a^{p,1-p}$ of total degree 1 in $C^*(\mathcal{U}, \mathrm{Hom}^*(\xi,\xi))$ satisfying

(1) $a_\alpha^{0,1} = a_\alpha$ (the differential on $\xi_\alpha^\cdot$) and

(2) $\delta a + a.a = 0.$

(We will explain the meaning of the equation (2) in the next section.)

In the above we have used local resolutions by holomorphic sheaves. Thus we will call the twisting cochain defined above a *holomorphic* twisting cochain. The above may be carried through just the same for local resolutions by sheaves of Σ_{U_α} modules, in which case we say that the twisting cochain is *smooth* .

0.B. TWISTED RESOLUTIONS.

Now we suppose that S is a sheaf of $\mathcal{O}_X$ (resp Σ_X) modules on X.

0.B.1. Definition.

A holomorphic (resp. smooth) *twisted resolution* of S is a triple $(\mathcal{U},\xi,a)$ satisfying the following conditions.

(i) $\mathcal{U}$ is a locally finite open cover of X. For the holomorphic case, we further require that $\mathcal{U}$ is a cover by Stein open sets

(ii) $\xi = (\xi^{\bullet}_{\alpha}, a_{\alpha})$ is a collection of local resolutions of S over each U_{α} by locally free $\mathcal{O}_{U_{\alpha}}$ (resp. $\Sigma_{U_{\alpha}}$) modules. We further require that ξ be of globally bounded length - that is, length $(\xi^{\bullet}_{\alpha}) < N$ for some N, all α.

(iii) a is a holomorphic (resp. smooth) twisting cochain over S - that is, we have the commutative diagram.

$$\begin{tikzcd}
\xi \arrow[r, "a^{1,0}"] \arrow[d] & \xi \arrow[d] \\
S \arrow[r, equal] & S
\end{tikzcd}$$

We further require that on degenerate simplices of the form $(\alpha_o,\ldots,\alpha_p)$ with $\alpha_i = \alpha_{i+1}$ for some i $(0 \leq i < p)$ a satisfies the following conditions.

$$a^{1,0} = 1, \qquad a^{p,1-p} = 0 \quad \text{for } p > 1.$$

To explain what this definition means, we look at the equation $\delta a + a.a = 0$. The first term is

$$a^{1,0}_{\alpha\beta}\ a^{0,1}_{\beta} - a^{0,1}_{\alpha}\ a^{1,0}_{\alpha\beta} = 0,$$

that is, it just says that $a^{1,0}$ is a chain map. The second term is

$$a^{1,0}_{\alpha\beta}\ a^{1,0}_{\beta\gamma} - a^{1,0}_{\alpha\gamma} + a^{0,1}_{\alpha}\ a^{2-1}_{\alpha\beta\gamma} + a^{2,-1}_{\alpha\beta\gamma}\ a^{0,1}_{\alpha} = 0.$$

That is, $a^{2,-1}_{\alpha\beta\gamma}$ is a chain homotopy for $a^{1,0}_{\alpha\beta}\ a^{1,0}_{\beta\gamma} - a^{1,0}_{\alpha\gamma}$. Thus in the twisting cochain each $a^{p+1,-p}$ is a 'correction term' or homotopy for $a^{p,1-p}$. The notion of twisting cochain is therefore a generalization of the familiar idea of transition functions for vector bundles. We use it to express the global interaction between our local resolutions $\xi_\alpha^\bullet$.

In the paper [6] we have the following result.

O.B.2. Theorem. (Toledo-Tong).

Suppose that S is a coherent analytic sheaf on X. Then there exists a holomorphic twisted resolution for S. Moreover, we may choose $(\mathcal{U},\xi,a)$ to satisfy

(a) length $(\xi_\alpha^\bullet) \leq \dim_{\mathbb{C}} X$ for all α

The proof is given in [6]. Basically we choose a Stein cover $\mathcal{U}$ over which S has local resolutions ξ and then proceed inductively to construct $a^{1,0}$ over the identity map on X, $a^{2,-1}$ over $a^{1,0}$, etc. The condition (a) follows from the Hilbert Syzygy Theorem.

Moreover, we note that the following form of equivalence holds.

O.B.3. Theorem (O'Brian [9]).

Suppose that S is a sheaf of $\mathcal{O}_X$ (resp Σ_X) modules, and that $(\mathcal{U},\xi,a)$, $(\mathcal{V},\mu,b)$ are two (holomorphic or smooth) twisted resolutions of S. Then there is a (holomorphic or smooth) twisted resolution $(\mathcal{U} \amalg \mathcal{V}, \xi \amalg \mu, c)$ where $\mathcal{U} \amalg \mathcal{V}$ denotes the disjoint union of $\mathcal{U}$ and $\mathcal{V}$, $\xi \amalg \mu$ denotes ξ on $\mathcal{U}$, μ on $\mathcal{V}$, and c is a twisting cochain which reduces to a, b when its indices are restricted to $\mathcal{U},\mathcal{V}$.

The proof (see [9]) is by induction just as that of O.B.2.

Finally, we note that since Σ_X is $\mathcal{O}_X$-flat, if $(\mathcal{U},\xi,a)$ is a holomorphic twisted resolution of S then tensoring with Σ_X gives us a smooth twisted resolution of $S \otimes_{\mathcal{O}_X} \Sigma_X$.

0.C. SIMPLICIAL METHODS (see [11,12])

0.C.1. Definition.

The simplicial category Δ. Δ has for objects the finite totally ordered sets $\Delta_n = [0,\ldots,n]$.

For morphisms it has the order-preserving maps

$$f\colon \Delta_n \to \Delta_m \quad (\text{that is, } x \le y \text{ implies } f(x) \le f(y)).$$

In particular we have the face maps

$$\delta_i : \Delta_n \to \Delta_{n+1} \quad (0 \le i \le n+1),$$

(the order-preserving injections such that $i \notin \delta_i/\Delta_n$)., and the degeneracy maps

$$S_i : \Delta_{n+1} \to \Delta_n \quad (0 \le i \le n),$$

(the order-preserving surjections such that $S_i(i) = S_i(i+1)$). These morphisms generate the morphisms of Δ. We shall say that a morphism of Δ is *of face type* if it is a composition of face maps, and of *degeneracy type* if it is a composition of degeneracy maps.

O.C.2. Definitions.

Given an arbitrary category C, a *simplicial object* in C is a contravariant functor $\Delta \to C$. A *morphism* of simplicial objects is a morphism of functors $\Delta \to C$. Suppose that $S \in Ob(C)$. Then we may define the *constant* simplicial object $S^{\cdot}$ with each $S^n = S$ and $\delta_i = S_i = id$. A simplicial object *over* S is a simplicial object $X^{\cdot}$ in C with a morphism $X^{\cdot} \to S^{\cdot}$.

Now suppose that $X^{\cdot}$ is a *simplicial space* (that is, a simplicial object in the category of topological spaces). We want a notion of sheaf on $X^{\cdot}$. First we need a definition.

O.C.3. Definition.

Suppose $u:X \to Y$ is a morphism of topological spaces, F a sheaf on X, G a sheaf on Y.

The set $Hom_u\ (G,F)$ of *u-morphisms* $G \to F$ is the set $Hom\ (u^*G,F) \approx Hom\ (G,\ \mu_*F)$.

O.C.4. Definition.

A *sheaf* $F_{\bullet}$ on the simplicial space $X^{\cdot}$ is

(a) a family of sheaves F_n on each X^n,

(b) for morphisms $f:\Delta_n \to \Delta_m$ of Δ, an $X^{\cdot}$f-morphism $F_{\bullet}(f):F_n \to F_m$ satisfying

$$F_{\bullet}(fog) = F_{\bullet}(f)oF_{\bullet}(g)$$

Now if $X^{\cdot}$ is a *simplicial manifold* (a simplicial object in the category of smooth manifolds), the sheaves $\Sigma_X p$ may be fitted together to form a sheaf on $X^{\cdot}$ by the pull-back morphisms. Thus we obtain a sheaf $\Sigma_{X^{\cdot}}$ on $X^{\cdot}$, the sheaf of germs of smooth functions.

Similarly, for a *simplicial analytic manifold* $X^{\cdot}$, we construct the sheaf $\mathcal{O}_{X^{\cdot}}$ of germs of holomorphic functions.

0.C.5. Definition.

The geometric n-simplex $|\Delta_n|$ is the subset of $\mathbb{R}^{n+1}$ given by the convex hull of the canonical base vectors. That is, $|\Delta_n| = \{(t_o,\dots t_n) \mid t_i \geq 0, \Sigma t_i = 1\}$.

Note that maps $f:\Delta_n \to \Delta_m$ extend by linearity to maps $|f| : |\Delta_n| \to |\Delta_m|$.

If M is a smooth manifold, we will denote by $A^r(M)$ the complex-valued smooth differential forms of degree r on M. Now suppose that $X^{\cdot}$ is a simplicial manifold. Then for each morphism $f: \Delta_p \to \Delta_q$ of Δ, we have a smooth map $X^{\cdot}f:X^q \to X^p$, and hence a pull-back of differential forms $(X^{\cdot}{}_f)^*:A^r(X^p) \to A^r(X^q)$. Similarly, we have a pull-back morphism

$$|f|^*: A^r(|\Delta_q|) \to A^r(|\Delta_p|).$$

Now we make the following definition.

0.C.6. Definition. (see [12]).

A *simplicial r-form* on $X^{\cdot}$ is a collection $\{C_p\}$ of forms $C_p \in A^r(X^p \times |\Delta_p|)$ such that for each morphism $f: \Delta_p \to \Delta_q$ of *face type*,

$$(1 \times |f|^*)C_q = ((X^{\cdot}f)^* \times 1)C_p \in A^r(X^q \times |\Delta_p|)$$

The set of such forms is denoted $A^r(X^{\cdot})$. The exterior derivatives d on each $X^p \times |\Delta_p|$ induce

$$d: A^r(X^{\cdot}) \to A^{r+1}(X^{\cdot}).$$

[We may restrict our attention to maps of face type, for the complex defined in this way is chain-equivalent to that defined using all Δ-morphisms. This is shown in Dupont [12]].

O.D. THE NERVE OF A COVER

Suppose that X is a paracompact smooth manifold, and $\mathcal{U}$ a locally-finite open cover.

O.D.1. Definition.

The nerve of $\mathcal{U}$, denoted $X^{\cdot}_{\mathcal{U}}$ is a simplicial manifold defined as follows. Suppose $\mathcal{U}$ is indexed as $\{U_\alpha\}_{\alpha\in I}$.

$$X^n_{\mathcal{U}} = \coprod_{\substack{(\alpha_o,\ldots,\alpha_n)\text{s.t.}\\ U_{\alpha_o,\ldots\alpha_n}\neq\emptyset}} U_{\alpha_o,\ldots,\alpha_n}$$

If δ_i and S_i are defined as in O.C.1, we may define

$$X^{\cdot}_{\mathcal{U}}(\delta_i) : X^{n+1}_{\mathcal{U}} \to X^n_{\mathcal{U}}, \quad U_{\alpha_o,\ldots,\alpha_{n+1}} \subset U_{\alpha_o,\ldots\hat{\alpha}_i,\ldots\alpha_{n+1}}.$$

$$X^{\cdot}_{\mathcal{U}}(S_i) : X^n_{\mathcal{U}} \to X^{n+1}_{\mathcal{U}}, \quad U_{\alpha_o,\ldots\alpha_i\ldots\alpha_n} \subset U_{\alpha_o,\ldots\alpha_i,\alpha_i,\ldots\alpha_n}$$

Then $X^{\cdot}_{\mathcal{U}}$ is a simplicial manifold *over* X in the sense of O.C.2.

Let $A^q(V)$ denote the complex-valued smooth q-forms on the open set $V \subset X$. Then we may define the well-known double complex.

O.D.2. Definition.

$$C^p(\mathcal{U},A^q) = \prod_{\substack{(\alpha_o,\ldots\alpha_p)\text{s.t}\\ U_{\alpha_o,\ldots\alpha_p}\neq\emptyset}} A^q(U\alpha_o,\ldots\alpha_p)$$

The differentials of this complex are

$$d: C^p(\mathcal{U},A^q) \to C^p(\mathcal{U},A^{q+1}) \text{ (exterior derivative)}$$

$$\delta: C^p(\mathcal{U},A^q) \to C^{p+1}(\mathcal{U},A^q) \text{ (Cech coboundary).}$$

From this we have the total complex $\mathrm{Tot}(C^*(\mathcal{U},A^*))$ with differential $\delta \pm d$. The cohomology of this complex is $H^*(X,\mathbb{C})$ (see [12]).

In the previous section we defined the De Rham complex for a simplicial space. Thus in particular we have the De Rham complex $A^*(X^{\bullet}_{\mathcal{U}})$ on the nerve $X^{\bullet}_{\mathcal{U}}$. Now we have the operation of *fibre integration* (see [12].). This gives us a map

$$\int_{|\Delta_p|} : A^q(U_{\alpha_o,\ldots\alpha_p}\times|\Delta_p|) \to A^{q-p}(U_{\alpha_o,\ldots,\alpha_p})$$

satisfying Stokes' Theorem

$$d \circ \int_{|\Delta_p|} = \int_{|\Delta_p|} \circ\, d + \int_{\partial|\Delta_p|}.$$

This says exactly that fibre integration defines a *chain map*

$$\int_{|\Delta|} : A^*(X^{\bullet}_{\mathcal{U}}) \to \mathrm{Tot}(C^*(\mathcal{U},A^*)).$$

It is shown in [12] that this chain map induces an isomorphism of cohomology.

Finally in this section (for want of anywhere else to put it), we recall the familiar 'gluing lemma' for sheaves. Suppose that on each open set U_α of our cover $\mathcal{U}$ we have defined a sheaf F_α. Suppose further that over each intersection $U_{\alpha\beta}$, we have isomorphisms

$$A_{\beta\alpha}: F_\alpha|_{U_{\alpha\beta}} \to F_\beta|_{U_{\alpha\beta}}$$

satisfying the 'cocycle condition' on $U_{\alpha\beta\gamma}$

$$A_{\gamma\beta}\, A_{\beta\alpha} = A_{\gamma\alpha}.$$

Then in these circumstances we have the lemma.

O.D.3. Lemma.

There is a sheaf F on X, and over each open set U_α an isomorphism $A_\alpha : F|U_\alpha \to F_\alpha$, such that on $U_{\alpha\beta}$ we have the relation $A_\beta = A_{\beta\alpha}\, A_\alpha$.

O.E. CONNECTIONS AND CURVATURE.

Holomorphic Connections.

Let Ω^r denote the sheaf of germs of analytic r-forms on the complex manifold X. Let S be any sheaf of $\mathcal{O}_X$-modules.

O.E.1. Definition. The jet-sequence of S.

The jet-sequence is the short-exact sequence

$$0 \to \Omega^1 \otimes S \to J^1(S) \to S \to 0,$$

where $J^1(S)$ is defined as follows. As a sheaf of $\mathbb{C}$-modules,

$J^1(S) = (\Omega^1 \otimes S) \oplus S$. The $\mathcal{O}_X$-action is given by

$$f.(w \otimes s,t) = (f.w \otimes s + df \otimes t, f.t)$$

For properties of the jet-sequence, see [1]. In particular we note that J^1 is additive and exact.

O.E.2. Definition.

A *holomorphic connection* for S is a splitting of the jet-sequence O.E.1.

O.E.3. Lemma.

If S is a locally free sheaf and X a Stein manifold, S has a holomorphic connection.

Proof.

By virtue of Cartan's Theorem B, any short exact sequence of locally free sheaves on a Stein manifold splits.

O.E.4. Definition.

Suppose that $(\mathcal{U},\xi,a)$ is a holomorphic twisted resolution. Then a *holomorphic connection* on $(\mathcal{U},\xi,a)$ is a holomorphic connection for each ξ^i_α. By virtue of O.E.3, any holomorphic twisted resolution has a holomorphic connection.

SMOOTH CONNECTIONS AND CURVATURE. (see [4,5])

Let X be a smooth manifold, $A^r(X)$ the complex-valued smooth r-forms on X. Let V be a smooth complex vector bundle on X. Then we give the traditional definition of a smooth connection on V.

O.E.5. Definition.

A smooth connection on V is a complex-linear map $\nabla:\Gamma V \to A^1(X) \otimes \Gamma V$ satisfying

$$\nabla(fs) = df \otimes s + f.\nabla s \quad \text{for } s \in \Gamma V,\ f \text{ a } C^\infty \text{ function.}$$

[Note that if X were a complex-analytic manifold and V a holomorphic vector bundle, and V had a holomorphic connection in the sense of O.E.1, then tensoring with the C^∞ functions and taking sections gives a smooth connection in the sense above.]

It is well-known that any vector-bundle has a smooth connection.

O.E.6. Definition.

If $(\mathcal{U},\xi,a)$ is a smooth twisted resolution, then a smooth connection for $(\mathcal{U},\xi,a)$ is a connection ∇^i_α for the vector bundle associated to each ξ^i_α. Hence every smooth twisted resolution has a smooth connection. Moreover, from the note above, if $(\mathcal{U},\xi,a)$ is a holomorphic twisted resolution with holomorphic connection ∇, then tensoring with Σ_X and taking sections gives a smooth connection on the associated smooth twisted resolution.

Now suppose V is a smooth complex vector bundle with connection ∇. Then if we take a local basis of sections $\{S_j\}$ for V in some neighbourhood, ∇ is given by a matrix of 1-forms w according to the formula

$$f^j S_j \to (df^k + f^j w_j^k) S_k$$

Then we define the *curvature* of ∇ with respect to this frame as the matrix of 2-forms $dw + w \wedge w$. If we change frames by an automorphism $g: V \to V$, and our connection has the matrix w' in the new frame, then

$$dw' + w' \wedge w' = g(dw + w \wedge w)g^{-1}.$$

0.E.7. Definition.

$K(\nabla) \in A^2(X) \otimes \mathrm{Hom}(V,V)$ is the globally defined 2-form given in a local frame by

$$dw + w \wedge w.$$

A connection ∇ is said to be *flat* if $K(\nabla) = 0$.

Now suppose that X has dimension $2n$, and ∇ is a connection for the vector bundle V. Then we may define the total Chern form.

0.E.8. Definition.

$c(V) \in A^*(X)$ is given by $\det(I + K(\nabla))$. We put $c(V) = 1 + c_1(V) + \dots + c_r(V)$, where $C_j(V)$ is a $2j$-form.

0.E.9. Lemma.

(1) $c(V)$ is a closed form, that is, $dc(V)$ is 0.

(2) If ∇' is another connection for V, and we put
$\det(I + K(\nabla')) = C'(V)$
then there is a σ in $A^*(X)$ such that
$d\sigma = C(V) - C'(V)$.

Thus $C(V)$ determines a cohomology class in $H^*(X,\mathbb{C})$, under the De Rham isomorphism.

(3) Let $\sigma_j(V)$ denote the topological Chern class of V in $H^{2j}(X,\mathbb{C})$. Then

$$(i/2\pi)^j \, c_j(V) = \sigma_j(V).$$

Proof.

All of these assertions are standard, and may be found in [5].

Now we wish to extend these results to complexes. Let $0 \to V_o \to \dots \to V_r \to 0$ be a complex of vector bundles. Choose a connection ∇_i for each V_i.

O.E.10. Definition.

The total Chern form of the complex $V_{\bullet}$ is given by

$$c(v_{\bullet}) = \mathop{}_{i=0}^{r} [c(v_i)]^{(-1)^i}$$

Then just as in Lemma O.E.9, $c(V_{\bullet})$ defines a cohomology class in $H^*(X,\mathbb{C})$, and for the component $c_j(V_{\bullet})$ in dimension 2j, we have

$$(i/2\pi)^j \, c_j(V_{\bullet}) = \sigma_j(V_{\bullet}),$$

where $\sigma_j(V_{\bullet})$ denotes the topological Chern class of the virtual bundle $\Sigma(-1)^i V_i$.

O.E.11. Lemma.

Let the complex $V_{\bullet}$ have differential D. Then the connections ∇_i for V_i are said to be *compatible* if $D\nabla_i = \nabla_{i+1}D$, that is, the ∇_i commute with the differential. Now suppose that the complex $V_{\bullet}$ is *exact*, and the connections ∇_i are compatible. Then

$$C(V_{\bullet}) = 1.$$

Proof. See [4].

1. CONSTRUCTING A RESOLUTION.

Suppose that S is a sheaf of $\mathcal{O}_X$(resp. Σ_X) modules on X, and that $(\mathcal{U},\xi,a)$ is a holomorphic (resp. smooth) twisted resolution of S. Recall that length $(\xi^\cdot)$ is globally bounded. Then by adding on zero modules if necessary, we may suppose that all our resolutions have the form

$$0 \to \xi_\alpha^{-n} \to \ldots \to \xi_\alpha^{o} \to S \to 0.$$

We are going to construct a resolution of the pulled-back sheaf S. on $X_{\mathcal{U}}^\cdot$. We shall give a special case of this construction first, as an example.

1.1. EXAMPLE.

Suppose that our twisted resolution is of length 1. Then each resolution has the form

$$0 \to \xi_\alpha^{-1} \to \xi_\alpha^{o} \to S|U_\alpha \to 0.$$

We will construct the resolutions on $X_{\mathcal{U}}^1$ and $X_{\mathcal{U}}^2$. On $X_{\mathcal{U}}^o$, our resolution is simply $\xi_\alpha^\cdot$ when restricted to each patch $U_\alpha \subset X_{\mathcal{U}}^o$.

So, on a typical patch $U_{\alpha\beta} \subset X_{\mathcal{U}}^1$, we want resolutions $(\alpha\beta)\xi_\alpha^\cdot$, $(\alpha\beta)\xi_\beta^\cdot$ and an isomorphism of resolutions

$$A_{(\alpha\beta)}^{\beta\alpha} : (\alpha\beta)\xi_\alpha^\cdot \to (\alpha\beta)\xi_\beta^\cdot .$$

These are given below:

$$\begin{tikzcd}[column sep=large, row sep=huge]
(\alpha\beta)\xi^{\cdot}_{\alpha} : 0 \arrow[r] & \xi^{-1}_{\alpha} \oplus \xi^{o}_{\beta} \arrow[r, "{\begin{bmatrix} a_{\alpha} & 0 \\ 0 & 1 \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} a_{\alpha}, & -a_{\alpha\beta} \\ a_{\beta\alpha}, & a_{\beta\alpha\beta} \end{bmatrix}}", "A^{\beta\alpha}_{(\alpha\beta)}"'] & \xi^{o}_{\alpha} \oplus \xi^{o}_{\beta} \arrow[r] \arrow[d, "{\begin{bmatrix} 1, & -a_{\alpha\beta} \\ a_{\beta\alpha}, & a_{\beta}\, a_{\beta\alpha\beta} \end{bmatrix}}"] & S|U_{\alpha\beta} \arrow[r] & 0 \\
(\alpha\beta)\xi_{\beta} : 0 \arrow[r] & \xi^{o}_{\alpha} \oplus \xi^{-1}_{\beta} \arrow[r, "{\begin{bmatrix} 1 & 0 \\ 0 & a_{\beta} \end{bmatrix}}"'] & \xi^{o}_{\alpha} \oplus \xi^{o}_{\beta} \arrow[r] & S|U_{\alpha\beta} \arrow[r] & 0
\end{tikzcd}$$

On a typical patch $U_{\alpha\beta\gamma} \subset X^2_{\mathcal{U}}$, we want resolutions $(\alpha\beta\gamma)\xi^{\cdot}_{\alpha}$, $(\alpha\beta\gamma)\xi^{\cdot}_{\beta}$, $(\alpha\beta\gamma)\xi^{\cdot}_{\gamma}$, and isomorphisms

$$\begin{tikzcd}
(\alpha\beta\gamma)\xi^{\cdot}_{\alpha} \arrow[rr, "A^{\beta\alpha}_{(\alpha\beta\gamma)}"] & & (\alpha\beta\gamma)\xi^{\cdot}_{\beta} \arrow[dl, "A^{\gamma\beta}_{(\alpha\beta\gamma)}"] \\
& (\alpha\beta\gamma)\xi^{\cdot}_{\gamma} \arrow[ul, "A^{\gamma\alpha}_{(\alpha\beta\gamma)}"] &
\end{tikzcd}$$

making the diagram commute. The resolutions are given as follows:

$$(\alpha\beta\gamma)\xi^{\cdot}_{\alpha} : 0 \to \xi^{-1}_{\alpha} \oplus \xi^{o}_{\beta} \oplus \xi^{o}_{\gamma} \longrightarrow \xi^{o}_{\alpha} \oplus \xi^{o}_{\beta} \oplus \xi^{o}_{\gamma} \to S \to 0$$

$$(\alpha\beta\gamma)\xi^{\cdot}_{\beta} : 0 \to \xi^{o}_{\alpha} \oplus \xi^{-1}_{\beta} \oplus \xi^{o}_{\gamma} \longrightarrow \xi^{o}_{\alpha} \oplus \xi^{o}_{\beta} \oplus \xi^{o}_{\gamma} \to S \to 0$$

$$(\alpha\beta\gamma)\xi^{\cdot}_{\gamma} : 0 \to \xi^{o}_{\alpha} \oplus \xi^{o}_{\beta} \oplus \xi^{-1}_{\gamma} \longrightarrow \xi^{o}_{\alpha} \oplus \xi^{o}_{\beta} \oplus \xi^{o}_{\gamma} \to S \to 0$$

We shall only write down the isomorphism $A^{\beta\alpha}_{(\alpha\beta\gamma)}$ because of space. The others are similar. In dimension -1

$$A^{\beta\alpha}_{(\alpha\beta\gamma)} = \begin{bmatrix} a_\alpha & -a_{\alpha\beta} & -a_{\alpha\gamma} \\ a_{\beta\alpha} & a_{\beta\alpha\beta} & a_{\beta\alpha\gamma} \\ 0 & 0 & 1 \end{bmatrix}$$

In dimension 0

$$A^{\beta\alpha}_{(\alpha\beta\gamma)} = \begin{bmatrix} 1 & -a_{\alpha\beta} & -a_{\alpha\gamma} \\ a_{\beta\alpha} & a_\beta\ a_{\beta\alpha\beta} & a_\beta\ a_{\beta\alpha\gamma} \\ 0 & 0 & 1 \end{bmatrix}$$

Now note that if we take the obvious inclusion maps

$$(\alpha\beta)^{\xi^\cdot_\alpha} \to (\alpha\beta\gamma)^{\xi_\alpha}, \quad (\alpha\beta)^{\xi_\beta} \to (\alpha\beta\gamma)^{\xi^\cdot_\beta},$$

then we have the commutative diagram

$$\begin{tikzcd} (\alpha\beta)^{\xi^\cdot_\alpha} \arrow[r, "A^{\beta\alpha}_{(\alpha\beta)}"] \arrow[d] & (\alpha\beta)^{\xi^\cdot_\beta} \arrow[d] \\ (\alpha\beta\gamma)^{\xi^\cdot_\alpha} \arrow[r, "A^{\beta\alpha}_{(\alpha\beta\gamma)}"] & (\alpha\beta\gamma)^{\xi^\cdot_\beta} \end{tikzcd}.$$

This is the data we will need to construct a resolution on $X^\cdot_{\mathcal{U}}$.

Now we go back to the general construction. Our strategy will be to construct on each patch $U_{\alpha_0 \ldots \alpha_p}$ of the nerve, resolutio

$F^{\alpha_i}_{\alpha_0,\ldots,\alpha_p}$ $(i = 0,\ldots p)$ of $S|U_{\alpha_0\ldots\alpha_p}$, connected by commuting isomorphisms of resolutions. These $F^{\alpha_i}_{\alpha_0 \ldots \alpha_p}$ will be constructed by adding to the local resolutions $\xi^{\cdot}_{\alpha_i}$ elementary sequences in the $\xi^{j}_{\alpha_k}$ $(k = 0,\ldots p)$, and the isomorphisms will be matrix functions of our twisting cochain. We may then use the 'gluing lemma' 0.0.3 to define a common resolution $F^{\cdot}_{\alpha_0 \ldots \alpha_p}$. By aggregation we therefore have a resolution on each $X^{p}_{\mathcal{U}}$ by locally free sheaves

$$0 \to F^{0}_{p} \to F^{1}_{p} \to \ldots \to F^{n}_{p} \to S_{p} \to 0$$

(where S_p denotes the natural pull-back of S to $X^{p}_{\mathcal{U}}$).

The next stage will be to show that the resolutions $F^{\cdot}_{p} \to S_{p} \to 0$ fit together to give a resolution by sheaves on $X^{\cdot}_{\mathcal{U}}$

$$F^{\cdot}_{\bullet} \to S_{\bullet} \to 0$$

That is, for each morphism $f\colon \Delta_p \to \Delta_q$, we must construct an $X^{\cdot}_{f}$- morphism of resolutions $F^{\cdot}_{p} \to F^{\cdot}_{q}$. We will further show that for the face map $\delta_i : \Delta_n \to \Delta_{n+1}$ the morphism $F^{\cdot}_{\bullet}\delta_i : (X^{\cdot}\delta_i)^{*} F^{\cdot}_{n} \to F^{\cdot}_{n+1}$ is injective, and that the co kernel is an elementary sequence. Similarly, for the degeneracy map $S_i\colon \Delta_{n+1} \to \Delta_n$ the morphism $F^{\cdot}_{\bullet}S_i : (X^{\cdot}S_i)^{*} F^{\cdot}_{n+1} \to F^{\cdot}_{n}$ is surjective, and its kernel is an elementary sequence.

CONSTRUCTING THE RESOLUTIONS $F^{\alpha_i}_{\alpha_o\ldots\alpha_p}$

Over the patch $U_{\alpha_o\ldots\alpha_p}$ we have (after restriction) the local resolutions

$$0 \to \xi^n_{\alpha_i} \to \ldots \to \xi^o_{\alpha_i} \to S \to 0$$

connected by the twisting cochain a (with its indices restricted to $\{\alpha_o\ldots\alpha_p\}$, and domains to $U_{\alpha_o\ldots\alpha_p}$). For this phase of the construction, we will drop the suffix α. Thus $\xi^{\cdot}_{\alpha_i}$ will be denoted by $\xi^{\cdot}_i$, etc. We are going to successively modify the $\xi^{\cdot}_i$ to make them isomorphic, by adding elementary sequences. We will call (a,ξ) $({}^oa,{}^o\xi)$.

Now to construct $({}^1a,{}^1\xi)$ from $({}^oa,{}^o\xi)$. We define ${}^1\xi$ as follows:

(a) $\quad {}^1\xi_i{}^j \quad = \quad {}^o\xi_i{}^{j-1}$ $(j < 0,\ j > 1)$

(b) for $j = 0,1$ we add an elementary sequence to give

$${}^1\xi_i{}^j = {}^o\xi^o_o \oplus {}^o\xi^o_1 \oplus \ldots \oplus {}^o\xi^{j-1}_i \oplus \ldots \oplus {}^o\xi^o_p$$

Now for the twisting cochain ${}^1a$. In dimensions $j \neq 0,1$ ${}^1a$ is just ${}^oa$ applied to dimension . In dimension 0 (we use block matrix notation)

$${}^1a_i^{0,1} = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \vdots & & & & & \\ 0 & \cdot\cdot & {}^oa_i & & \cdot & 0 \\ 0 & \cdot & \cdot & & & 1 \end{bmatrix}$$

$$[{}^1a^{1,0}_{ji}]^s_r \qquad \begin{array}{ll} \delta_{rs} & (r \neq i,j) \\ {}^oa_i & (r = i,\ s = i) \\ -{}^oa_{is} & (r = i,\ s \neq i) \\ {}^oa_{ji} & (r = j,\ s = i) \\ {}^oa_{jis} & (r = j, s \neq i) \end{array}$$

(r denotes rows, s columns)

${}^1a^{q,1-q}_{j_o,..j_q}$ $(q \geq 2)$ is given by

$$((-1)^{q+1}\ {}^oa^{q+1,-q}_{j_o,..jq,0},\ (-1)^{q+1}\ {}^oa^{q+1,-q}_{j_o,..j_q,1}, \ldots,\ a^{q,1-q}_{j_o,\ldots jq};..(-1)^{q+1}a^{q+1,-q}_{j_o,..j}$$

In dimension 1

$$[{}^1a^{1,0}_{ji}]^s_r \quad = \quad \begin{array}{ll} \delta_{rs} & (r \neq i,j) \\ 1 & (r = i,\ s = i) \\ -{}^oa_{is} & (r = j,\ s \neq i) \\ {}^oa_j\ {}^oa_{jis} & (r = j,\ s \neq i) \end{array}$$

${}^1a^{q,1-q} = 0$ for $q > 1$.

Then we have the following lemma.

1.2. Lemma.

$$\delta\,{}^1a + {}^1a \cdot {}^1a = 0$$

For the proof (which is a long matrix calculation) see appendix.

Thus we have constructed $({}^1\xi, {}^1a)$ from $({}^o\xi, {}^oa)$. For the r-th step of the induction we simply replace 1,0 by r,r-1 in the above. So we may proceed until we have constructed $({}^n\xi, {}^na)$. Then ${}^n\xi^j = 0$ for $j < 0$. Thus our resolutions have the form

$$0 \to {}^n\xi_i^0 \to {}^n\xi_i^1 \to \quad \dots \to {}^n\xi_i^n \to S \to 0$$

and ${}^n a$ has the property

$${}^n a^{q,1-q} = 0 \text{ for } q > 1.$$

Hence the ${}^n a_{ji}^{1,0}$ are isomorphisms of resolutions, and we have

$${}^n a_{kj}^{1,0} \quad {}^n a_{ji}^{1,0} = {}^n a_{ki}^{1,0} \quad .$$

At this point we will change our notation. We put ${}^n\xi_i = F^{\alpha_i}_{\alpha_0 \dots \alpha_p}$, ${}^n a_{ji}^{1,0} = A^{\alpha_j \alpha_i}_{\alpha_0 \dots \alpha_p}$. Then the lemma O.D.3. allows us to glue these sheaves together with the isomorphisms, and we obtain the resolution $F^\bullet_{\alpha_0 \dots \alpha_p}$.

CONSTRUCTING FACE AND DEGENERACY MAPS.

The space $X^p_{\mathcal{U}}$ is a disjoint union of patches of the form $U_{\alpha_0 \dots \alpha_p}$. Thus by the previous construction, we obtain a resolution

$$0 \to F^0_p \to F^1_p \to \dots \to F^n_p \to \to S_p \to 0$$

on each $X^p_{\mathcal{U}}$. For each morpism $f: \Delta_p \to \Delta_q$ we must now construct an $X^\bullet_f$-morphism of resolutions

$$F^\bullet_\bullet f : F^\bullet_p \to F^\bullet_q \ .$$

It is sufficient to construct $F^\bullet_\bullet \delta_i$ and $F^\bullet_\bullet s_i$, where δ_i and s_i are the canonical face and degeneracy operators.

We must then show that these maps satisfy the appropriate relations to generate the morphism of Δ.

First, let us look at $\delta_i: \Delta_{p-1} \to \Delta_p$ $(i = 0,\ldots p)$. We restrict to a typical patch $U_{\alpha_0\ldots\alpha_p} \subset X^p_U$. Here we must construct a morphism

$$F^{\cdot}\delta_i: F^{\cdot}_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_p}|U_{\alpha_0\ldots\alpha_p} \to F^{\cdot}_{\alpha_0\ldots\alpha_p}.$$

Thus for $j \neq i$ we must construct morphisms over $U_{\alpha_0\ldots\alpha_p}$

$$F^{\alpha_j}_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_p} \to F^{\alpha_j}_{\alpha_0\ldots\alpha_p}$$

such that the diagram below commutes (for $j, k \neq i$).

$$\begin{tikzcd}
F^{\alpha_j}_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_p} \arrow[r] \arrow[d, "A^{\alpha_k\alpha_j}_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_p}"'] & F^{\alpha_j}_{\alpha_0\ldots\alpha_p} \arrow[d, "A^{\alpha_k\alpha_j}_{\alpha_0\ldots\alpha_p}"] \\
F^{\alpha_k}_{\alpha_0\ldots\hat{\alpha}_k\ldots\alpha_p} \arrow[r] & F^{\alpha_k}_{\alpha_0\ldots\alpha_p}
\end{tikzcd}$$

We do this inductively. So we return to the notation of the previous section. The r-th modification of our complexes over $(\alpha_0\ldots\alpha_p)$ is denoted by $({}^r\xi, {}^ra)$ as before. On the simplex $(\alpha_0,\ldots\hat{\alpha}_i\ldots\alpha_p)$ we denote the r-th modification by $({}^r_i\xi, {}^r_ia)$.

Now clearly for $r = 0$ we have over $U_{\alpha_0\ldots\alpha_p}$ ${}^o_i\xi_j = {}^o\xi_j$ for $j \neq i$ and the diagram

$$\begin{tikzcd} {}^{o}_{i}\xi \arrow[r] \arrow[d, "{}^{o}_{i}a"'] & {}^{o}\xi \arrow[d, "{}^{o}a"] \\ {}^{o}_{i}\xi \arrow[r] & {}^{o}\xi \end{tikzcd}$$

Commutes for each bit of the twisting cochain.

Next we write down the morphisms for r = 1. For each $j \neq i$, we want a morphism ${}^{1}_{i}\xi^{\cdot}_{j} \to {}^{1}\xi^{\cdot}_{j}$. In dimensions other than 0,1 it is the identity. For d = 0,1 it is given by

$$\begin{array}{c} {}^{o}_{i}\xi^{o}_{o} \oplus \dots \oplus {}^{o}_{i}\xi^{o}_{i-1} \oplus {}^{o}_{i}\xi^{o}_{i+1} \oplus \dots \oplus {}^{o}_{i}\xi^{d}_{j} \oplus \dots \oplus {}^{o}_{i}\xi^{o}_{p} \\ \big\downarrow \\ {}^{o}_{i}\xi^{o}_{o} \oplus \dots \oplus {}^{o}\xi^{o}_{i-1} \oplus {}^{o}\xi^{o}_{i} \oplus {}^{o}\xi^{o}_{i+1} \oplus \dots \oplus {}^{o}\xi^{d}_{j} \oplus \dots \oplus {}^{o}\xi^{o}_{p} \end{array}$$

$$(x_o, \dots x_{i-1}, x_{i+1}, \dots x_j, \dots x_p) \to (x_o, \dots x_{i-1}, 0, x_{i+1}, \dots, x_j, \dots x_p)$$

Once again we have the commutative diagram

$$\begin{tikzcd} {}^{1}_{i}\xi \arrow[r] \arrow[d, "{}^{1}_{i}a"'] & {}^{1}\xi \arrow[d, "{}^{1}a"] \\ {}^{1}_{i}\xi \arrow[r] & {}^{1}\xi \end{tikzcd}$$

Note in particular that the cokernel of the map ${}^{1}_{i}\xi^{\cdot}_{j} \longrightarrow {}^{1}\xi^{\cdot}_{j}$ $(j \neq i)$ is an elementary sequence (in this case the sequence $0 \to \xi^{o}_{i} \to \xi^{o}_{i} \to 0$) and that in the diagram

$$\begin{array}{ccccccccc}
0 & \to & {}^{1}_{i}\xi^{\cdot}_{j} & \to & {}^{1}\xi^{\cdot}_{j} & \to & \mathrm{cok} & \to & 0 \\
 & & \Big\downarrow {}^{1}_{i}a^{1,0}_{kj} & & \Big\downarrow {}^{1}a^{1,0}_{kj} & & \Big\| & & \\
0 & \to & {}^{1}_{i}\xi^{\cdot}_{k} & \to & {}^{1}\xi^{\cdot}_{k} & \to & \mathrm{cok} & \to & 0
\end{array}$$

the identity is induced on the cokernel.

As in the previous section, we continue inductively by simply replacing 1,0 with r,r-1. Hence putting r=n we have, as advertised, constructed morphisms $F^{\alpha_j}_{\alpha_o\ldots\hat{\alpha}_i\ldots\alpha_p} \longrightarrow F^{\alpha_j}_{\alpha_o\ldots\alpha_p}$ $(j \neq i)$ over $U_{\alpha_o\ldots\alpha_p}$, commuting with the isomorphisms. Hence we have constructed the $X^{\cdot}\delta_i$-morphism $F^{\cdot}_{\cdot}\ \delta_i : F^{\cdot}_{p-1} \to F^{\cdot}_{p}$. This morphism is injective, and we have the short exact sequence

$$0 \to (X^{\cdot}\delta_i)^{*}\ F^{\cdot}_{p-1} \to F^{\cdot}_{p} \to \mathrm{cok}\ (F^{\cdot}_{\cdot}\delta_i) \to 0$$

where $\mathrm{cok}(F^{\cdot}_{\cdot}\delta_i)$ is an elementary sequence.

Now for the degeneracy morphism $S_i : \Delta_{p+1} \to \Delta_p$ $(0 \leq i \leq p)$. Here we look at a typical inclusion $U_{\alpha_o,\ldots,\alpha_i,\ldots\alpha_p}$ $U_{\alpha_o,\ldots\alpha_i,\ldots\alpha_p}$ Once again we will denote the r-th modification over $(\alpha_o,\ldots,\alpha_p)$ by $({}^{r}\xi,{}^{r}a)$, and that over $(\alpha_o,\ldots\alpha_i,\alpha_i,\ldots\alpha_p)$ by $({}^{r}_{i}\xi,{}^{r}_{i}a)$. We have the identity morphism ${}^{o}_{i}\xi \to {}^{o}\xi$.

For r = 1 we again have the identity morphism except in dimensions d = 0,1. Here we distinguish two cases.

(1) ${}^1_i\xi_j \to {}^1\xi_j (j \neq i)$. The morphism here is

$$\begin{array}{c} {}^o_i\xi^o_o \oplus {}^o_i\xi^o_i \oplus \dots \oplus {}^o_i\xi^d_j \oplus \dots \oplus {}^o_i\xi_i \oplus {}^o_i\xi_i \oplus \dots \oplus {}^o_i\xi_p \\ \downarrow \\ {}^o_i\xi^o_o \oplus \dots \oplus {}^o\xi^d_j \oplus \dots \oplus {}^o\xi^o_i \oplus \dots \oplus {}^o\xi_p \end{array}$$

$$(x_o, \dots x_j, \dots x_i, x_i' \dots x_p) \to (x_o, \dots, x_j, \dots, x_i + x_i', \dots x_p)$$

(2) ${}^1_i\xi_j \to {}^1\xi_j (j = i)$. There are of course two resolutions in ${}^1_i\xi$ for $j = i$, but they are isomorphic, so we just give one of the morphisms.

$${}^o_i\xi^o_o \oplus \dots \oplus {}^o_i\xi^d_i \oplus {}^o_i\xi^o_i \oplus \dots \oplus {}^o_i\xi^o_p$$

$$(x_o, \dots, x_i, x_i', \dots x_p) \to (x_o, \dots, x_i, \hat{x}_i', \dots x_p)$$

As before we have the commutative diagram

$$\begin{array}{ccc} {}^1_i\xi & \to & {}^1\xi \\ {}^1_ia \downarrow & & \downarrow {}^1a \\ {}^1_i\xi & \to & {}^1\xi \end{array}$$

and we accomplish the inductive step by replacing 1,0 with r, r-1. Clearly, putting r=n, the morphism ${}^n_i\xi \to {}^n\xi$ is surjective, and the $a^{1,0}$ induce the identity on the kernel, which is an elementary sequence.

Hence we have constructed the $X^{\cdot}S_i$-morphism

$$F^{\cdot}_{\cdot} S_i : (X^{\cdot}S_i)^* F^{\cdot}_{p+1} \to F^{\cdot}_p$$

This morphism is surjective, and the kernel is an elementary sequence.

Finally, we must check that the face and degeneracy operators we have constructed satisfy the appropriate relations. These relations are given in [12].

1.3. Lemma.

(1) $F^{\cdot}_{\cdot}(\delta_j)\, F^{\cdot}_{\cdot}(\delta_i) = F^{\cdot}_{\cdot}(\delta_i)\, F^{\cdot}_{\cdot}(\delta_{j-1})$ $i < j$

(2) $F^{\cdot}_{\cdot}(S_j)\, F^{\cdot}_{\cdot}(S_i) = F^{\cdot}_{\cdot}(S_i)\, F^{\cdot}_{\cdot}(S_{j+1})$ $i \leq j$

(3) $F^{\cdot}_{\cdot}(S_j)\, F^{\cdot}_{\cdot}(\delta_i) = F^{\cdot}_{\cdot}(\delta_i)\, F^{\cdot}_{\cdot}(S_{j-1})$ $i < j$

id $i = j,\ j + 1$

$F^{\cdot}_{\cdot}(\delta_{i-1})\, F^{\cdot}_{\cdot}(S_j)$ $i > j+1$

Proof.

We note that since we have commuting isomorphisms, it is sufficient to check these equations for one of our resolutions. We shall check for the first modification. This is simply a formal question, so we look just at the form of the maps. Thus $F^{\cdot}_{\cdot}(\delta_i)$ has the form

$$(x_0,\ldots,x_p) \to (x_0,\ldots,x_{i-1},0,x_i,\ldots,x_p)$$

If on $U_{\alpha_0\ldots\alpha_{p+1}}$ we choose the resolution $F^{\alpha_0}_{\alpha_0\ldots\alpha_{p+1}}$ and assume $j \neq 0$, then we may write $F^{\cdot}_{\cdot}(S_j)$

$$(x_0,\ldots,x_{p+1}) \to (x_0,\ldots,x_{j-1},x_j+x_{j+1},\ldots,x_{p+1})$$

But now these maps have the same form as the maps

$$|\delta_i| : |\Delta_p| \to |\Delta_{p+1}| ,$$

$$|s_j| : |\Delta_{p+1}| \to |\Delta_p|$$

and since these maps satisfy the relations above (see [12]) our lemma is proved.

1.4. SUMMARY

From a holomorphic (resp. smooth) twisted resolution $(\mathcal{U},\xi,a)$ we have constructed a resolution on $X^{\bullet}_{\mathcal{U}}$ by sheaves of $\mathcal{O}_{X^{\bullet}_{\mathcal{U}}}$ (resp. $\Sigma^{\bullet}_{X_{\mathcal{U}}}$ modules,

$$0 \to F^{o}_{\bullet} \to F^{1}_{\bullet} \to .. \to F^{n}_{\bullet} \to S_{\bullet} \to 0$$

with the following properties.

(1) On each $X^{p}_{\mathcal{U}}$, the sheaves F_p are locally free.

(2) $F^{\bullet}_{o}|U_{\alpha} = \xi^{\bullet}_{\alpha}$. (up to numbering)

(3) If $f:\Delta_p \to \Delta_q$ is a map of face type, then

$F^{\bullet}_{\bullet}f: (X^{\bullet}f)^{*}F^{\bullet}_{p} \to F^{\bullet}_{q}$ is injective, and $\operatorname{cok}(F^{\bullet}_{\bullet}f)$ is an elementary sequence.

(4) If $f: \Delta_p \to \Delta_q$ is a map of degeneracy type, then

$F^{\bullet}_{\bullet}f: (X^{\bullet}f)^{*} F^{\bullet}_{p} \to F^{\bullet}_{q}$ is surjective, and $\ker (F^{\bullet}_{\bullet}f)$ is elementary.

Now suppose that we have a simplex $\alpha = (\alpha_o, \ldots \alpha_p)$. If β is a sub-simplex of α, then we will write $\beta \leq \alpha$. Then the properties above imply the following.

(5) If $\beta \leq \alpha$, then $F^{\cdot}_{\alpha} \simeq F^{\cdot}_{\beta} \oplus E^{\beta}_{\alpha}$, where E^{β}_{α} is an elementary sequence in the $\xi^{\cdot}_{\alpha_j}$, $j = 0, \ldots p$.

(6) If $\gamma \leq \beta \leq \alpha$, then $E^{\gamma}_{\alpha} = E^{\gamma}_{\beta} \oplus E^{\beta}_{\alpha}$.

(7) If $\gamma \leq \beta \leq \alpha$, then we have the commutative diagram (we leave out restriction signs) over U_{α}

$$\begin{tikzcd}
0 \arrow[r] & F^{\cdot}_{\beta} \arrow[r] & F^{\cdot}_{\alpha} \arrow[r] & E^{\beta}_{\alpha} \arrow[r] & 0 \\
0 \arrow[r] & F^{\cdot}_{\gamma} \oplus E^{\gamma}_{\beta} \arrow[r] \arrow[u, "\wr"] & F^{\cdot}_{\gamma} \oplus E^{\gamma}_{\alpha} \arrow[r] \arrow[u, "\wr"] & E^{\beta}_{\alpha} \arrow[r] \arrow[u, equal, "id"] & 0
\end{tikzcd}$$

where the bottom map is induced by the natural inclusion $E^{\gamma}_{\beta} \to E^{\gamma}_{\alpha}$ from (6).

2. THE CHERN CLASSES

For the moment we will work exclusively in the smooth case. So let $(\mathcal{U},\xi,a)$ be a smooth twisted resolution of S, and suppose that it has a connection ∇. That is, we have a smooth connection on each ξ_α^{-j}. Now suppose that we have an elementary sequence E in the ξ_α^{-j}. Then the direct sum connections give compatible connections (0.E.11) for E.

Now as in (1.4), let

$$0 \to \mathcal{F}^{0}_{\bullet} \to \dots \to \mathcal{F}^{n}_{\bullet} \to S_{\bullet} \to 0$$

be the resolution on $X^{\bullet}_{\mathcal{U}}$ constructed from $(\mathcal{U},\xi,a)$. Clearly on a patch $U_\alpha \subset X^{0}_{\mathcal{U}}$, we have connections for the $\mathcal{F}^{\bullet}_{\alpha}$. Now consider a typical patch $U_{\alpha_0,\dots\alpha_p} \subset X^{p}_{\mathcal{U}}$. From (1.4(5)) we have isomorphisms over $U_{\alpha_0\dots\alpha_p}$

$$\mathcal{F}^{\bullet}_{\alpha_0\dots\alpha_p} \simeq \mathcal{F}^{\bullet}_{\alpha_j} \oplus E^{\alpha_j}_{\alpha_0\dots\alpha_p} \quad (j = 0,\dots p),$$

where $E^{\alpha_j}_{\alpha_0\dots\alpha_p}$ is an elementary sequence in the ξ's. Thus taking the direct sum, ∇ induces a connection on the right-hand side, and so we have connections ∇_{α_j} for $\mathcal{F}^{\bullet}_{\alpha_0\dots\alpha_p}$, $j = 0,\dots p$.

Moreover, from (1.4(7)) we have the diagram over $U_{\alpha_0\dots\alpha_p}$

2.1.
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}^{\bullet}_{\alpha_0\dots\hat{\alpha}_i\dots\alpha_p} \arrow[r] & \mathcal{F}^{\bullet}_{\alpha_0\dots\alpha_p} \arrow[r] & E^{\alpha_0\dots\hat{\alpha}_i\dots\alpha_p}_{\alpha_0\dots\alpha_p} \arrow[r] & 0 \\
& 0 \arrow[r] & \mathcal{F}^{\bullet}_{\alpha_j} \oplus E^{\alpha_0\dots\hat{\alpha}_i\dots\alpha_j\dots p}_{\alpha_0\dots\hat{\alpha}_i\dots\alpha_p} \arrow[r] \arrow[ul] & \mathcal{F}^{\bullet}_{\alpha_j} \oplus E^{\alpha_0\dots\hat{\alpha}_j\dots\alpha_p}_{\alpha_0\dots\alpha_p} \arrow[r] \arrow[ul] & E^{\alpha_0\dots\hat{\alpha}_i\dots\alpha_p}_{\alpha_0\dots\hat{\alpha}_p\dots\alpha_p} \arrow[r] \arrow[ul, equal, "id"'] & {}
\end{tikzcd}$$

On the bottom line we have for $j \neq i$ compatible connections induced by ∇. Thus the connections ∇_{α_j} on the top are compatible in each dimension. So on $X^p_{\mathcal{U}}$, we have defined connections ∇_j, $(j = 0,\ldots p)$ on $\mathcal{F}^{\cdot}_p$.

We wish to construct a total Chern form $C(S)$ in the simplicial De Rham complex $A^*(X^{\cdot}_{\mathcal{U}})$. (O.C.6). So let $t_o,\ldots t_p$ be barycentric co-ordinates for the geometric p-simplex $|\Delta_p|$. We may pull back the complex $\mathcal{F}^{\cdot}_p$ to the complex $\overline{\mathcal{F}}^{\cdot}_p$ on $X^p_{\mathcal{U}} \times |\Delta_p|$. We may define a connection ∇^r for each $\overline{\mathcal{F}}^r_p$

$$\nabla^r = \sum_{j=0}^{p} t_j \nabla^r_j.$$

Then we have a Chern form $C^p(S) \in A^*(X^p_{\mathcal{U}} \times |\Delta_p|)$,

$$C^p(S) = \prod_{r=0}^{n} [\det(I + K(\nabla^r))]^{(-1)^r}$$

2.2. <u>Lemma</u>.

The $C^p(S)$ define a total Chern form $C(S)$ in $A^*(X^{\cdot}_{\mathcal{U}})$.

<u>Proof</u>.

We need to show that for face map

$$\delta_i : \Delta_{p-1} \to \Delta_p,\ (1 \times |\delta_i|^*)C^p = ((X^{\cdot}\delta_i)^* \times 1)C^{p-1} \text{ in } X^p_{\mathcal{U}} \times |\Delta_{p-1}|.$$

On a typical patch $U_{\alpha_o\ldots\alpha_p}$, this means that we must show that

$$C^p(S) = C^{p-1}(S),$$

on the face $t_i = 0$ of $U_{\alpha_o\ldots\alpha_p} \times |\Delta_p|$.

This follows from the diagram 2.1, for we note that the identity is induced on the cokernel. Thus the connection on the cokernel for each $j \neq i$ is the same. So on the face $t_i = 0$ of $U_{\alpha_0 \ldots \alpha_p} \times |\Delta_p|$, the connection on the cokernel is given by

$\sum_{j \neq i} t_j \cdot \nabla_E = \nabla_E$, where ∇_E is the connection induced by ∇ on the elementary sequence. So we have a compatible sequence of connections for the cokernel. Then let

$$C_E = \prod_{r=0}^{n} [\det(I + K(\nabla_E^r))]^{(-1)^r}$$

By Lemma O.E.11, we have on the face $t_i = 0$

$$C^p(S) = C^{p-1}(S) \wedge C_E,$$

and $C_E = 1$. Thus we have a well-defined total Chern form $C(S)$ in $A^*(X_{\mathcal{U}}^{\cdot})$

Clearly $C(\nabla)$ is a closed form, since it is closed on each $X_{\mathcal{U}}^p \times |\Delta_p|$ by O.E.9. Recall that in O.D. we noted the existence of the chain equivalence

$$\int_{|\Delta|} : A^*(X_{\mathcal{U}}^{\cdot}) \to \mathrm{Tot}[C^*(\mathcal{U}, A^*)]$$

2.3. <u>Definition</u>.

Let w be a form of degree m on $X_{\mathcal{U}}^r \times |\Delta_r|$. We shall say that w is of *type* (p,q) if w has the form

$$w = \sum_{0 \le i_1 < \ldots < i_q \le r} n_{i_1, \ldots, i_q} dt_{i_1} \wedge \ldots \wedge dt_{i_q}$$

where $n_{i_1 \ldots i_q}$ is the pull-back of a form on $X_{\mathcal{U}}^r$ of degree p.

We may therefore expand any form w by type

$$w = w^{0,m} + w^{1,m-1} + \dots + w^{m,a}$$

Note that if $w^{p,q}$ has $q < r$, then $\int_{|\Delta_r|} w^{p,q} = 0$. Thus

$$\int_{|\Delta_r|} w = \sum_{q \geq r} \int_{|\Delta_r|} w^{m-q,q}$$

Now look at the form $C^{2p}(\nabla) \in A^{2p}(X^{\cdot}_{\mathcal{U}})$.

The expansion of $C^{2p}(\nabla)$ by type on any $X^r_{\mathcal{U}} \times |\Delta_{\mathcal{U}}|$ has non-zero terms only for (s,t) with $s \geq t$. Hence

$$\int_{|\Delta|} C^{2p}(\nabla) \in \mathrm{Tot}^{2p}\,(C^*(\ ,A^*)$$

has the form

$$C^{0,2p} + C^{1,2p-1} + C^{2,2p-2} + \dots + C^{p,p}$$

and since it is a cocycle, we have

$$\delta C^{p,p} = 0.$$

Now let us return to the analytic case. Suppose that $(\mathcal{U},\xi,a)$ is a holomorphic twisted resolution, and ∇ a holomorphic connection. Then by tensoring with Σ_X, we obtain the associated smooth twisted resolution and connection. So we once again obtain the Chern form

$$C(\nabla) \in A^*\,(X^{\cdot}_{\mathcal{U}}).$$

But now if we look at the form $C^{2p}(\nabla)$ on $X^p_{\mathcal{U}} \times |\Delta_p|$, the component of type (p,p) is given by

$$n \quad dt_1 \wedge \dots \wedge dt_p,$$

where η is the pull-back of a holomorphic form on $X^p_{\mathcal{U}}$.

Thus in the holomorphic case we have

$$\bar{\partial}\, C^{p,p} = 0,$$

and so $C^{p,p}$ is a p-cocycle in the Cech complex $C^*(\mathcal{U},\Omega^p)$. Now we have, for any open cover $\mathcal{U}$,

$$H^*\{\mathrm{Tot}[C^*(\mathcal{U},A^*)]\} \simeq H^*(X,\mathbb{C})$$

and if $\mathcal{U}$ is a Stein cover

$$H^*(\mathcal{U},\Omega^p) \simeq H^*(X,\Omega^p).$$

So we may state the theorem.

2.4. <u>Theorem</u>.

Let S be a coherent analytic sheaf on X. Then there are:

(1) Chern classes $C_p(S) \in H^{2p}(X,\mathbb{C})$, depending only on the smooth equivalence class of S,

(2) Atiyah-Chern classes $C_p^A(S) \in H^p(X,\Omega^p)$ depending only on the analytic equivalence class of S.

There is a Stein cover $\mathcal{U}$ for which we may choose a representative of $C_p(S)$ as a total p-cocycle in $C^*(\mathcal{U},A^*)$,

$$C^{p,p} + C^{p-1,p+1} + \ldots + C^{0,2p},$$

such that $C^{p,p}$ is a representative for $C_p^A(S)$ in $C^*(\mathcal{U},\Omega^p)$

<u>Proof</u>.

To prove the theorem, we must show that the classes in 1 and 2 are independent of the choice of twisted resolution and connection.

So suppose that $(\mathcal{U},\xi,a)$ and $(\mathcal{V},\mu,b)$ are two smooth twisted resolutions of S, with connections $\nabla_{\mathcal{U}}$, $\nabla_{\mathcal{V}}$ respectively. The process above defines total Chern forms $C_{\mathcal{U}} \in \mathrm{Tot}(C^*(\mathcal{U},A^*))$ and $C_{\mathcal{V}} \in \mathrm{Tot}(C^*(\mathcal{V},A^*))$.

We shall show that under the natural restriction maps, $C_{\mathcal{U}}$ and $C_{\mathcal{V}}$ are cohomologous in $\mathrm{Tot}[C^*(\mathcal{U} \cap \mathcal{V},A^*)]$. By Theorem O.B.3, there is a smooth twisted resolution $(\mathcal{U} \amalg \mathcal{V}, \xi\amalg\mu, C)$, where C is a twisting cochain reducing to a on $\mathcal{U}$, b on $\mathcal{V}$. Now $\nabla_{\mathcal{U}}$ and $\nabla_{\mathcal{V}}$ define a connection for this twisted resolution. So as in Section 1, we may construct the associated resolution on $X^{\bullet}_{\mathcal{U} \amalg \mathcal{V}}$. Note that on 'pure' subsets $U_{\alpha_0 \ldots \alpha_p}$, $V_{\beta_0 \ldots \beta_q}$, this resolution is just the one previously constructed on $X^{\bullet}_{\mathcal{U}}$ (resp. $X^{\bullet}_{\mathcal{V}}$).

Thus if $C_{\mathcal{U} \amalg \mathcal{V}}$ denotes the total Chern form in $\mathrm{Tot}[C^*(\mathcal{U}\amalg\mathcal{V},A^*)]$ then under the natural inclusions

$$\mathcal{U} \subset \mathcal{U} \amalg \mathcal{V} \quad , \quad \mathcal{V} \subset \mathcal{U} \amalg \mathcal{V},$$

$C_{\mathcal{U} \amalg \mathcal{V}}$ maps to $C_{\mathcal{U}}$, $C_{\mathcal{V}}$ respectively.

So in $\mathrm{Tot}(C^*(\mathcal{U} \cap \mathcal{V}, A^*))$ the forms defined by $C_{\mathcal{U}}$, $C_{\mathcal{V}}$ are given from $C_{\mathcal{U} \amalg \mathcal{V}}$ by two different refinement maps, and hence are cohomologous.

Exactly the same argument now shows that in the holomorphic case, the Atiyah Chern forms

$$C^A_{p,\mathcal{U}} \in C^p(\mathcal{U},\Omega^p), \; C^A_{p,\mathcal{V}} \in C^p(\mathcal{V},\Omega^p) \text{ are cohomologous in}$$

$C^*(\mathcal{U} \cap \mathcal{V},\Omega^p)$.

We shall now establish the basic properties of these classes.

2.5. Lemma.

If S is a locally free sheaf on X, then

(1) $C(S)$ is the classical total Chern class

(2) $C^A(S)$ is the total Atiyah Chern class.

Proof.

For a locally free sheaf S we may take local resolutions of length 0,

$$0 \to \xi_\alpha^0 \to S|U_\alpha \to 0,$$

and the $a_{\beta\alpha}^{1,0}$ are just the transition functions. In this case we just have the traditional construction (see [3]).

2.6. Lemma.

If S,T are coherent analytic sheaves on X, then

(1) $C(S \oplus T) = C(S) \wedge C(T)$ in $H^*(X,\mathbb{C})$

(2) $C^A(S \oplus T) = C^A(S) \wedge C^A(T)$ in $H^*(X,\Omega^*)$

Proof.

We take a Stein cover $\mathcal{U}$ over which both S,T have twisted resolutions. Then if $(\mathcal{U},\xi,a)$, $(\mathcal{U},\mu,b)$ are twisted resolutions for $S \oplus T$. If we choose connections for $(\mathcal{U},\xi,a)$, $(\mathcal{U},\mu,b)$ then the direct sum connection is defined on $(\mathcal{U}, \xi \oplus \mu, a \oplus b)$. Then the product formula over each $X_{\mathcal{U}}^p \times |\Delta_p|$ proves the lemma.

2.7. Lemma.

Suppose that X is compact. S is a coherent analytic sheaf on X. Let $\sigma_p(S) \in H^{2p}(X,\mathbb{Z})$ denote the classes of Atiyah-Hirzenbruch (see [2]). Then we have

$$(i/2\pi)^p c_p(S) = \sigma_p(S).$$

Proof.

The Atiyah-Hirzenbruch classes are defined in the compact case as follows. We tensor S with the sheaf of germs of real-analytic functions on X, R_X. Then by results of Grauert mentioned in [2], $S \otimes_{\mathcal{O}_X} R_X$ has a global resolution by locally-free R_X-modules,

$$0 \to V_o \to \ldots \to V_n \to S \otimes_{\mathcal{O}_X} R_X \to 0.$$

Then the Atiyah-Hirzenbruch class $\sigma_p(S)$ is defined to be the p-th Chern class of the virtual bundle $\Sigma(-1)^i \overline{V}_i$ where $\overline{V}_i$ is the topological vector bundle underlying the real-analytic sheaf V_i. But now (Lemma 0.1) we know that Σ_X is a flat R_X-module. So tensoring with Σ_X, we have a smooth twisted resolution of S, with the cover being the one open set X. Hence by Lemma 0.E.9,

$$(i/2\pi)^p c_p(S) = \sigma_p(S).$$

So we have shown that our constructions satisfy the correct formal properties for a generalization of the Chern and Atiyah classes to coherent sheaves, and that in the compact case they agree up to a constant factor with the classes of Atiyah-Hirzenbruch. We conclude with an example which though trivial does demonstrate the method of calculation.

EXAMPLE

Let $\mathbb{P}$ denote the complex projective 1-space. We will consider $\mathbb{P}$ as the Rieman sphere $\mathbb{C} \cup \{\infty\}$. Let $\mathcal{O}$ denote the sheaf of germs of holomorphic functions, and J the sheaf of ideals of the subvariety $\{0\}$. We wish to calculate the Chern class of the coherent sheaf $S = \mathcal{O}/J$. The stalks of this sheaf are zero except at the origin, where the sheaf has stalk $\mathbb{C}$. Let $U_\alpha = \mathbb{P} - \{\infty\}$, $U_\beta = \mathbb{P} - \{0\}$. This is a Stein cover of $\mathbb{P}$. Let z be the complex co-ordinate for $\mathbb{C}$.

Over U_α we have the resolution

$$\xi^{\bullet}_{\alpha}:\ 0 \to \mathcal{O} \xrightarrow[f \to z.f]{} \mathcal{O} \to S \to 0$$

Over U_β we may choose

$$\xi^{\bullet}_{\beta}:\ 0 \to 0 \to 0 \to S \to 0$$

Then to construct a twisting cochain, we must give chain maps $a_{\beta\alpha}$, $a_{\alpha\beta}$ and chain homotopies $a_{\beta\alpha\beta}$, $a_{\alpha\beta\alpha}$. Clearly $a_{\beta\alpha}$, $a_{\alpha\beta}$ must be the zero map, as must $a_{\beta\alpha\beta}$. $a_{\alpha\beta\alpha}$ is to satisfy the twisting cochain equation

$$a_{\alpha\beta}a_{\beta\alpha} + a_{\alpha}\ a_{\alpha\beta\alpha} = 1,$$

and hence must be given by $f \to f/z$ (on $U_{\alpha\beta}$).

Then our resolutions over $U_{\alpha\beta}$ are given by (see Example (1.1))

$$\begin{tikzcd}
(\alpha\beta)\xi^{\bullet}_{\alpha}: \arrow[d, "A^{\alpha\beta}_{(\alpha\beta)}"'] & 0 \arrow[r] & \mathcal{O} \arrow[r, "f \mapsto z.f"] \arrow[d, "f \mapsto z.f"'] & \mathcal{O} \arrow[r] \arrow[d, "\mathrm{id}"] & S \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
(\alpha\beta)\xi^{\bullet}_{\beta}: & 0 \arrow[r] & \mathcal{O} \arrow[r, "\mathrm{id}"] & \mathcal{O} \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

Then $c_1^A(S)$ is given by Tr $\{A^{\alpha\beta}_{(\alpha\beta)} dA^{\beta\alpha}_{(\alpha\beta)}\}$ and thus $c_1^A(S) = -dz/z \in C^1(\mathcal{U},\Omega^1)$.

APPENDIX

Proof of Lemma 1.2.

$$\delta\, {}^1a + {}^1a \,.\, {}^1a = 0$$

We need to check four things.

(1) ${}^1a,{}^{1,0}$ is a chain map in dimensions 0,1.

(2) In dimension 1

$${}^1a^{1,0}_{kj} \,.\; {}^1a^{1,0}_{j1} = {}^1a^{1,0}_{ki}$$

(3) In dimension 0

$${}^1a^{1,0}_{kj} \quad {}^1a^{1,0}_{ji} + {}^1a^{0,1}_{k} \quad {}^1a^{2,-1}_{kji} = {}^1a^{1,0}_{ki}$$

(4) In dimension 0, the component of $\delta\,{}^1a + {}^1a\,.\,{}^1a$ is bidegree $(q,2-q)$ is zero for $q > 2$.

(1) We need to show that

$${}^1a^{0,1}_{j} \quad {}^1a^{1,0}_{ji} = {}^1a^{1,0}_{ji} \quad {}^1a^{0,1}_{i}$$

The left-hand side is

$$\begin{array}{cc}
 & \begin{array}{ccccc} & & j & & \end{array} \\
\begin{array}{c} \\ \\ j \\ \\ \end{array} &
\begin{bmatrix} 1, & 0 & \dots & 0, & 0 \\ 0, & 1, & \dots & & 0 \\ 0 & & a_j, & \dots & 0 \\ 0 & & & & 1 \end{bmatrix}
\end{array}
\quad
\begin{array}{cc}
 & \begin{array}{ccccc} \phantom{-a_{i0},} & & i & \phantom{a_{i,i+1}} & \phantom{a_{ip}} \end{array} \\
\begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} &
\begin{bmatrix} 1, \; 0 & & & & 0 \\ -a_{i0}, -a_{i1} & \dots & ;\; a_i \,; & a_{i,i+1} \dots & a_{ip} \\ 0 & & & 1 & 0 \\ a_{ji0}, \; a_{ji1}, & \dots & ;\; a_{ji} \,; & a_{ji,i+1}, \dots & a_{jip} \\ 0 & & & & 1 \end{bmatrix}
\end{array}$$

which gives

$$
\begin{array}{cc}
\begin{matrix} \\ i \\ \\ j \\ \\ \end{matrix} &
\begin{bmatrix}
1 & & & 0 \\
-a_{i0}, -a_{i1}, \ldots & ; a_i ; a_{i,i+1}, \ldots & & a_{ip} \\
0 & 1 & & 0 \\
a_j a_{ji0}, a_j a_{ji1}, \ldots & ; a_j a_{ji} ; a_j a_{j,i,i+1}, & & a_j a_{jip} \\
0 & & & 1
\end{bmatrix}
\end{array}
$$

The right hand side is

$$
\begin{array}{cc}
& \begin{matrix} & i & \end{matrix} \\
\begin{matrix} \\ i \\ \\ j \\ \\ \end{matrix} &
\begin{bmatrix}
1, \; 0 & & 0 \\
-a_{i0}, \ldots ; 1 ; -a_{i,i+1}, \ldots & & -a_{ip} \\
0 & 1 & 0 \\
a_j a_{ji0}, \ldots ; a_{ji} ; a_j a_{ji,i+1} \cdot a_j a_{jip} & & \\
0 & & 1
\end{bmatrix}
\end{array}
\;
\begin{array}{cc}
& \begin{matrix} & i & \end{matrix} \\
\begin{matrix} \\ i \\ \\ \\ \\ \end{matrix} &
\begin{bmatrix}
1,0 & & 0 \\
0 & ; a_i ; 0 \ldots\ldots\ldots & 0 \\
 & & \\
 & & \\
0 & & 1
\end{bmatrix}
\end{array}
$$

which gives

$$
\begin{array}{cc}
& \begin{matrix} & i & \end{matrix} \\
\begin{matrix} \\ i \\ \\ j \\ \\ \end{matrix} &
\begin{bmatrix}
1, & & 0 \\
-a_{i0}, \ldots, ; a_i ; -a_{i,i+1}, \ldots & & -a_{ip} \\
0 & 1 & 0 \\
a_j a_{ji0}, \ldots ; & a_{ji} a_i ; & a_j a_{jip} \\
0 & & 1
\end{bmatrix}
\end{array} .
$$

Since in row j, column i we have $a_j a_{ji} = a_{ji} a_i$, the right-hand side is equal to the left-hand side.

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(2) We put $P = {}^1a^{1,0}_{kj}\ {}^1a^{1,0}_{ji}$, where P has elements P^s_r in row r, column s.

P is given by

$$
\begin{array}{cc}
 & \begin{array}{ccc} \quad & j & \qquad\qquad \end{array} \\
\begin{array}{c} \\ j \\ \\ k \\ \\ \end{array} &
\left[\begin{array}{lr}
1,\ldots & 0 \\
-a_{j0},\ldots;1; & -a_{jp} \\
 & 0 \\
a_k a_{kj0},\ldots;a_{kj};\cdot & a_k a_{kjp} \\
0 & 1
\end{array}\right]
\end{array}
\quad
\begin{array}{cc}
 & \begin{array}{ccc} \qquad & i & \qquad\qquad \end{array} \\
\begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} &
\left[\begin{array}{lr}
1, & 0 \\
-a_{i0},\ldots;1; & -a_{ip} \\
\quad 1 & 0 \\
a_j a_{ji0},\ldots;a_{ji}; & a_j a_{jip} \\
0 & 1
\end{array}\right]
\end{array}
$$

We consider first the case $k \neq i$.

Clearly for $r \neq i, j,k$ $\quad P^s_r = \delta_{rs}$

For $r=j$ and $s \neq i,j$ $\quad P^s_r = a_{ji}a_{is} + a_j a_{jis} - a_{js} = 0$

For $r=j$, $s=i$, $\quad P^s_r = -a_{ji} + a_{ji} = 0$

For $r=j$, $s=j$, $\quad P^s_r = a_{ji}a_{ij} + a_j a_{jij} = 1.$

Thus for $r \neq i,k$ $\quad P^s_r = \delta_{rs}$

For $r = i$, $s \neq i$, $\quad P^s_r = -a_{is}$

For $r = i$, $s = i$, $\quad P^s_r = 1$

For $r = k, s \neq i,j$, $\quad P^s_r = -a_k a_{kji} a_{is} + a_{kj} a_{jis} + a_k a_{kjs}$

$= a_k(a_{kj}a_{jis} + a_{kjs} - a_{kji}a_{is})$.

But the equation $\delta a + a.a = 0$ on the simplex (kjis) is

$-a_{kis} + a_{kjs} + a_{kj}a_{jis} - a_{kji}a_{is} = 0.$

Thus $\quad P^s_r = a_k a_{kis}$

For $r = k$, $s = j$, $\quad P^s_r = -a_k a_{kji} a_{ij} + a_{kj} a_j a_{jij}$

$= a_k(a_{kj}a_{jij} - a_{kji}a_{ij})$.

But since $a_{kjj} = 0$, the previous argument gives

$$P_r^s = a_k a_{kij}$$

For $r = k$, $s = i$, $P_r^s = a_k a_{kji} + a_{kj} a_{ji} = a_{ki}$

Thus we have shown that $P = {}^1a_{ki}^{1,0}$.

Now suppose $k = i$, so $P = {}^1a_{ij}^{1,0} \ {}^1a_{ji}^{1,0}$

For $r \neq i,j$, $P_r^s = \delta_{rs}$.

For $r = i$, $s \neq i,j$ $P_r^s = -a_i a_{iji} a_{is} + a_{ij} a_j a_{jis} + a_i a_{ijs}$.

So $P_r^s = a_i(-a_{iji} a_{is} + a_{ij} a_{jis} + a_{ijs}) = 0$.

For $r = i$, $s = j$, $P_r^s = -a_i a_{iji} a_{ij} + a_{ij} a_j a_{jij} = 0$.

For $r = i$, $s = i$, $P_r^s = a_i a_{iji} + a_{ij} a_{ji} = 1$

For $r = j$, $s \neq i,j$, $P_r^s = a_{ji} a_{is} + a_j a_{jis} - a_{js} = 0$

For $r = j$, $s = i$, $P_r^s = -a_{ji} + a_{ji} = 0$

For $r = j$, $s = j$ $P_r^s = a_{ji} a_{ij} + a_j a_{jij} = 1$.

Thus P is the identity, and we have checked (2).

(3) We want to check that

$${}^1a_{kj}^{1,0} \ {}^1a_{ji}^{1,0} + {}^1a_k^{0,1} \ {}^1a_{kji}^{2-1} = {}^1a_{ki}^{1,0} .$$

Let $Q = {}^1a_k^{0,1} \ {}^1a_{kji}^{2,-1}$. Then Q is given by

$$\begin{array}{cc} & \\ \\ \\ k & \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \cdot \begin{array}{ccccc} & & i & & \\ (-a^{3,-2}_{kjio}, & -a^{3,-2}_{kji1}, \ldots; & a^{2,-1}_{kji}; & \ldots & -a^{3,-2}_{kjip}) \end{array}$$

which is

$$\begin{array}{cc} & \begin{array}{ccc} \qquad\qquad\qquad & i & \qquad\qquad\qquad \end{array} \\ k & \begin{bmatrix} 0 & & 0 \\ 0 & 0 & 0 \\ -a_k a_{kjio}, -a_k a_{kji1}, & ; a_k a_{kji}; & -a_k a_{kjip} \\ 0 & & 0 \end{bmatrix} \end{array}$$

That is, Q^s_r is given as follows.

For $r \neq k$, $Q^s_r = 0$

For $r = k$, $s \neq i$, $Q^s_r = -a_k a_{kjis}$.

For $r = k$, $s = i$, $Q^s_r = -a_k a_{kji}$.

As before, we denote ${}^1a^{1,0}_{kj}$ ${}^1a^{1,0}_{ji}$ by $\{P^s_r\}$

For $r \neq i,j,k$, $P^s_r = \delta_{rs}$.

For $r = j, s \neq i,j$, $P^s_r = -a_{js} + a_{ji}a_{is} + a_j a_{jis} = 0$.

For $r = j$, $s = j$, $P^s_r = a_{ji}a_{ij} + a_j a_{jij} = 1$.

For $r = j$, $s = i$, $P^s_r = -a_{ji}a_i + a_j aj_i = 0$.

Thus for $r \neq i,k$ $P^s_r = \delta_{rs}$.

For $r = i$, $s \neq i$ $P^s_r = -a_{is}$

For $r = i$, $s = i$ $P^s_r = a_i$.

Thus for $r \neq k$, $\quad P_r^s = [{}^1a_{ki}^{1,0}]_r^s$.

For $r = k$, $s \neq i,j$, $P_r^s = a_{kjs} + a_{kj}\, a_{jis} - a_{kji}\, a_{is}$

For $r = k$, $s = i$ $\quad P_r^s = a_{kji}\, a_i + a_{kj}\, a_{ji}$

Now we want to show that

$$P_r^s + Q_r^s = [{}^1a_{ki}^{1,0}]_r^s \ .$$

This is true for $r \neq k$ by the above.

For $r = k$, $s \neq i,j$.

$$P_r^s + Q_r^s = a_{kjs} + a_{kj}a_{jis} - a_{kji}a_{is} - a_k a_{kjis} = a_{kis}$$

For $r = k$, $s = j$

$$P_r^s + Q_r^s = a_{kj}a_{jij} - a_{kji}a_{ij} - a_k a_{kjij} = a_{kik}$$

(since $a_{kjj} = 0$).

For $r = k$, $s = i$

$$P_r^s + Q_r^s = a_{kj}a_{ji} + a_{kji}a_i + a_k a_{kji} = a_{ki}.$$

Thus $P+Q = {}^1a_{ki}^{1,0}$.

(4) Finally, we want to show for $q > 2$

$$[\delta\, {}^1a]^{q,2-q} + {}^1a^{0,1} \cdot {}^1a^{q,1-q} + \ldots + {}^1a^{q-1,2-1} \cdot {}^1a^{1,0} + {}^1a^{q,1-q} \cdot {}^1a^{0,1} = 0$$

in dimension 0. The right-hand term vanishes since we have defined ${}^1a^{q,1-q}$ as 0 in dimension 1. We will evaluate the above expression on $(j_o,\ldots j_q)$

We look first at the term

$$ {}^{1}a^{q-1,2-q}_{u_o,\dots j_{q-1}} \cdot {}^{1}a^{1,0}_{j_{q-1}j_q} $$

given by

$$ \begin{array}{c} j_q \\ (a^{q,1-q}_{j_o,\dots j_{q-1},0},\dots;(-1)^q a^{q-1,2-q}_{j_o\dots j_{q-1}};\dots a^{q,1-q}_{j_o,\dots j_{q-1},p})^{j_q} \end{array} \begin{array}{c} \\ \\ \\ j_{q-1} \\ \\ \end{array} \begin{array}{c} j_q \\ \begin{bmatrix} 1, & 0 & & 0 & 0 \\ -a_{jq,o} & \dots\dots; & a_{jq}; & \dots\dots & -a_{jq,p} \\ 0 & & 1 & & 0 \\ a_{jqj_{q-1},}0,\dots; & & a_{j_{q-1}j_q}; & \dots & a_{j_qj_{q-1}p} \\ 0 & & & & 1 \end{bmatrix} \end{array} $$

This is a row vector R, say, with columns indexed by s. For $s \neq j_q$, $R^s = -a_{j_o,\dots j_{q-1},j_q} a_{j_q,s} + (1)^q a_{j_o\dots j_{q-1}} a_{j_qj_{q-1}s}$.

For $s = j_q$, $R^s = (-1)^q\, a_{j_o\dots j_{q-1}}\, a_{j_{q-1}j_q} + a_{j_o,\dots j_q,j_q}\, a_{j_q}$.

Now for the higher order terms

$$ {}^{1}a^{q-r,1-q+r} \cdot {}^{1}a^{r,1-r} \quad (r > 1), $$

the front term ${}^{1}a$ is equal to a. Then by adding up it is easy to see that the row-vector

$$ [\delta\,{}^{1}a + {}^{1}a.{}^{1}a]^{q,2-q}_{j_o\dots j_q} $$

is given in columns $s \neq j_q$ by

$$ [\delta a + a.a]^{q+1,2-q}_{j_o,\dots j_q,s} = 0, $$

and in column j_q by

$$[\delta a + a.a]^{q,2-q}_{j_o,\ldots j_q} = 0.$$

Hence we have finally shown

$$\delta\,{}^1a + {}^1a.{}^1a = 0.$$

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